On a Small Elliptic Perturbation of a Backward-Forward Parabolic Problem, with Applications to Stochastic Models

by

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Abstract

We consider an elliptic PDE in two variables. As one parameter approaches zero, this PDE collapses to a parabolic one, that is forward parabolic in a part of the domain and backward parabolic in the remainder. Such problems arise naturally in various stochastic models, such as fluid models for data-handling systems and Markov-modulated queues. We employ singular perturbation methods to study the problem for small values of the parameter.

1 Introduction

We consider the following boundary value problem for $F = F(x, \xi)$

$$DF_{xx} + (a-\xi)F_x + F_{\xi\xi} + (\xi F)_\xi = 0; \qquad x > 0, \quad -\infty < \xi < \infty \tag{1.1}$$

$$DF_x(0,\xi) + (a-\xi)F(0,\xi) = 0, \quad -\infty < \xi < \infty \tag{1.2}$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} F(x,\xi)dx\, d\xi = 1. \tag{1.3}$$

Here a and D are positive constants, and F is a bivariate probability density. This problem arose as an asymptotic limit of a Markov-modulated queueing model, where the input process

is generated by N sources that turn "on" and "off" at exponential waiting times. When on, a source generates a Poisson arrival stream to the queue. The joint distribution of the number of on sources and the queue length satisfies a complicated system of difference equations, that as $N \to \infty$ may be approximated by the problem (1.1)-(1.3). The variable x is related to the queue length and ξ corresponds to a scaled, centered measure of the number of on sources. An empty queue has $x = 0$, and $\xi = 0$ means that the number of active sources equals its mean value.

The diffusion coefficient D in (1.1) measures variability effects in the service time distribution, while a measures the difference between average output and input rates to the queue. The condition $a > 0$ guarantees stability and the existence of a steady state distribution. Note that $D \to 0$ means that the queue becomes a deterministic, or "fluid", process. A detailed derivation of (1.1)-(1.3) can be found in [1]. There we also showed that the solution can be reduced to either a Fredholm integral equation of the second kind (for $F(0, \xi)$), or to finding a single eigenvector of an infinite matrix, whose elements are expressed in terms of Laguerre functions.

The complexity of the exact solution to (1.1)-(1.3) suggests that an asymptotic analysis may be fruitful. In [1] we considered the limit $D \to \infty$ and discussed numerical methods that work well for D large and $D = O(1)$. The purpose of this note is to analyze the opposite limit, namely $D \to 0^+$. This is a very singular limit, as we show in what follows. Also, the numerical methods involve truncating an infinite matrix, say to an $M \times M$ matrix. For $D \to 0^+$ the eigenvector we seek decays on the scale (see [1, section 4.2]) $O(D^{-2})$. Thus for say $D = .1$, M must be of the order of about 500 before an accurate result is obtained. The basic matrix is not sparse; it does become diagonally dominant for $D \to \infty$, but not for $D \to 0^+$.

Let us briefly discuss the problem with $D = 0$. Denoting by $\mathcal{F}(x, \xi)$ the solution to (1.1) with $D = 0$, we see that the elliptic PDE degenerates into a parabolic one, that is forward parabolic in the range $\xi > a$ and backward parabolic for $\xi < a$. The boundary condition (1.2) becomes $\mathcal{F}(0, \xi) = 0$ and can only be applied in the range $\xi > a$, where the PDE is forward parabolic. The study of backward/forward parabolic problems dates as far back as 1914 (see Gerrey [2]) and particular problems of the type here are analyzed in [3,4,5]. In [4] we give an explicit expression for $\mathcal{F}(x, \xi)$, subject to the condition $\mathcal{F}(\infty, \xi) = (2\pi)^{-1/2} e^{-\xi^2/2}$ (replacing (1.3)). The function $\mathcal{F}$ can also be interpreted as an asymptotic limit of a probability distribution for a discrete stochastic model, that was formulated and analyzed in [6]. Here we are interested in how the solution of the elliptic PDE approaches that of the parabolic one. For the latter the values of $\mathcal{F}(0, \xi)$ for $\xi < a$ are unknown and must be computed as a part of the solution. Probabilistically, this corresponds to boundary mass along this part of the boundary. For any $D > 0$, (1.3) shows that F is a proper density function. Now the "no-flux" boundary condition (1.2) applies for all ξ, and no boundary mass develops. We also note that by integrating (1.1) over $x \in (0, \infty)$ and using (1.2), the marginal distribution

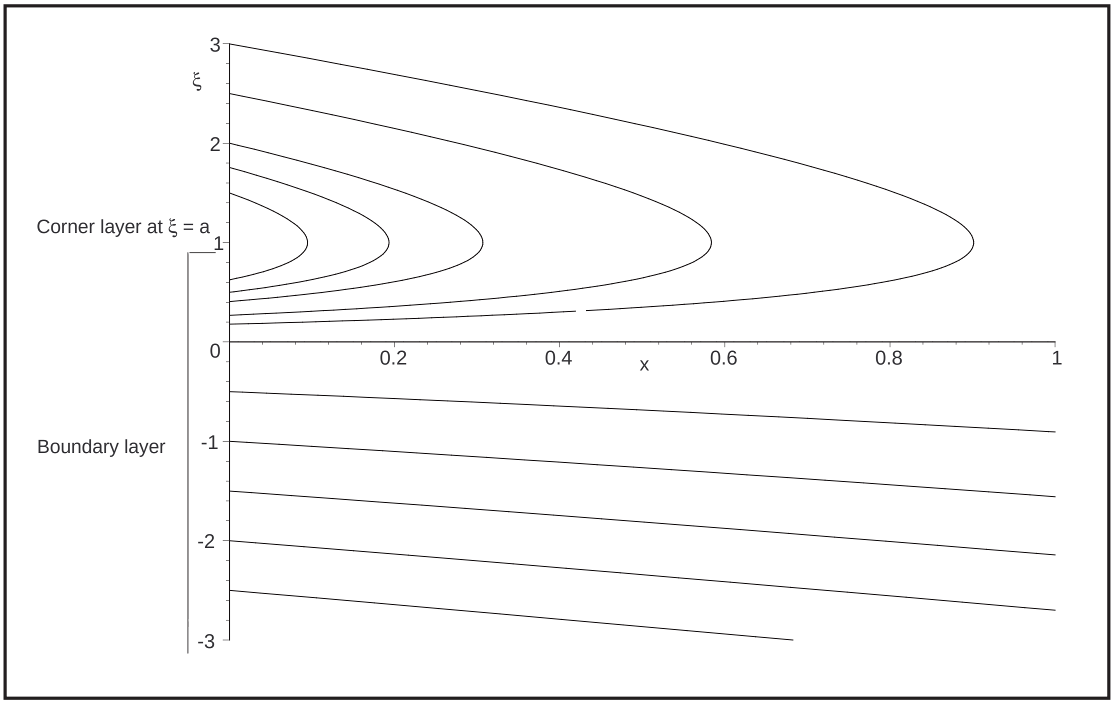

Figure 1: A sketch of the deterministic trajectories for $a = 1$.

$\int_0^\infty F(x, \xi)dx$ satisfies an elementary ODE that is readily solved to give

$$\int_0^\infty F(x, \xi)dx = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}. \tag{1.4}$$

Here we also used the normalization (1.3). Since F is a density in x while $\mathcal{F}$ is a distribution in x, we expect to compare F to $\mathcal{F}_x$ for $x > 0$.

To get a qualitative picture of the solution, we plot in Figure 1 the deterministic approximation to (1.1).

This corresponds to neglecting diffusion in both the x- and ξ-variables, and is the phase-plane flow $\dot{x} = \xi - a$, $\dot{\xi} = -\xi$. We plot a few trajectories for $x > 0$, and note that once they hit $x = 0$ (necessarily with $\xi < a$), they stay at $x = 0$ and flow to the equilibrium at $\xi = 0$. The figure also indicates the boundary and corner layers that arise in the analysis for $D \to 0$.

2 Singular Perturbation Analysis

We shall analyze (1.1)-(1.3) for $D \to 0^+$ in the three ranges (i) $x > 0$, $-\infty < \xi < \infty$ (or $x \approx 0$, $\xi > a$); (ii) $x \approx 0$, $\xi < a$; and (iii) $x \approx 0$, $\xi \approx a$.

2.1 Outer Solution

We assume $x > 0$ and expand $F(x,\xi)$ as $F(x,\xi) = F_0(x,\xi) + DF_1(x,\xi) + O(D^2)$ to find that

$$(\xi - a)F_{0,x} = F_{0,\xi\xi} + (\xi F_0)_\xi. \tag{2.1}$$

We note that when $\xi > a$, (2.1) is a diffusion equation with x taking the place of time, and when $\xi < a$ it is a diffusion equation run in reverse time. As is well-known [2,3,5], we need to impose boundary conditions for (2.1) on the "incoming" half of the boundary, where $x = 0$ and $\xi > a$. We can derive the appropriate "half boundary condition" for (2.1) by considering a boundary layer about $x = 0$.

We thus consider the boundary layer where $x = O(D)$. Setting $x = Du$ and $F(x,\xi) = G(u,\xi) \sim D^{\nu_0} G_0(u,\xi)$, we find that

$$G_{0,uu} + (a-\xi)G_{0,u} = 0; \quad u > 0, \qquad -\infty < \xi < \infty \tag{2.2}$$

$$G_{0,u}(0,\xi) + (a-\xi)G_0(0,\xi) = 0, \qquad -\infty < \xi < \infty \tag{2.3}$$

and hence G_0 is proportional to $\exp\big[(\xi - a)u\big]$. For $\xi > a$ this grows exponentially as $u \to \infty$ and cannot match to the outer solution $F_0(x,\xi)$ as $x \to 0^+$, for any choice of ν_0. We thus conclude that no boundary layer develops along $\xi > a$, where (2.1) is forward parabolic. Thus F_0 must satisfy $F_0(0,\xi) = 0$ for $\xi > a$, which is the limit of (1.2) as $D \to 0^+$.

The explicit solution to (2.1), subject to the above "half boundary condition," is given in [4]. There we solved (2.1) using a Laplace transform in the x-variable. Due to the forward/backward nature of the PDE, the spectrum has both positive and negative eigenvalues. Insisting that the solution decay as $x \to \infty$, the positive eigenvalues must be absent. This, together with the half boundary condition for $\xi > a$ is sufficient to determine the solution.

Noting that F_0 should correspond to $\mathcal{F}_x$, we give two different representations of this solution. The first is the contour integral

$$\begin{aligned} F_0(x,\xi) &= K\frac{1}{2\pi i}\int_{\mathrm{Br}} \frac{e^{-\xi^2/4}}{\sqrt{2\pi}} D_{a\theta+\theta^2}(\xi + 2\theta)A(\theta)e^{\theta x}\,d\theta, \qquad (2.4)\\ A(\theta) &= \frac{a}{\theta + a}\exp\left[-\theta\zeta\big(\tfrac{1}{2}\big) + \frac{1}{2}\gamma\theta(\theta + a)\right]\prod_{m=1}^{\infty}\frac{\delta_m}{\theta + \delta_m}\exp\left[\frac{\theta}{\sqrt{m}} - \frac{\theta(\theta+a)}{2m}\right],\\ \delta_n &= \frac{1}{2}\big[a + \sqrt{a^2 + 4n}\,\big], \quad n = 0,1,2,\ldots\,. \end{aligned}$$

Here $D_p(\cdot)$ is the parabolic cylinder function of order p, γ is Euler's constant and $\zeta(\cdot)$ is the Riemann zeta function. The Bromwich contour Br in (2.4) is any vertical contour in the half-plane $\mathrm{Re}(\theta) > -a$, where the integrand is analytic. Note that the only singularities of the integrand are simple poles at $\theta = -\delta_n$, $n \geq 0$. If we close the contour to the left, then we can convert the integral to the sum of the residues at the poles. This leads to a second representation of the solution in terms of the problem's eigenfunctions:

$$
\begin{aligned}
F_0(x,\xi) &= K\sum_{n=0}^{\infty} c_n e^{-\delta_n x}\frac{e^{-\xi^2/2}}{\sqrt{2\pi}}e^{\xi\delta_n}H_n\left(\frac{\xi-2\delta_n}{\sqrt{2}}\right), && (2.5)\\
c_n &= \frac{a2^{-n/2}e^{-\delta_n^2}}{a-\delta_n}\exp\left[\delta_n\zeta(\tfrac{1}{2})+\frac{1}{2}n\gamma\right]{\prod_{m=1}^{\infty}}'\frac{\delta_m}{\delta_m-\delta_n}\exp\left[-\frac{\delta_n}{\sqrt{m}}-\frac{n}{2m}\right],\\
&\quad n\geq 1,\\
c_0 &= ae^{-a^2}\exp\left[a\zeta(\tfrac{1}{2})\right]\prod_{m=1}^{\infty}\frac{\delta_m}{\delta_m-a}\exp\left(-\frac{a}{\sqrt{m}}\right).
\end{aligned}
$$

The product in the expression for c_n omits the term $m=n$ and $H_n(\cdot)$ is the n^{th} Hermite polynomial.

The constant K is a normalization constant that will ultimately be determined from (1.3). We cannot determine it at this stage since we shall show that there is also an $O(1)$ probability mass in a boundary layer near $x=0$ (with $\xi<a$), that contributes to (1.3).

2.2 Boundary Layer

While no boundary layer develops for $\xi>a$, there is one for $\xi-a<0$ and $x=O(D)$. We again scale $x=Du$, and set

$$F(x,\xi)=G(u,\xi)=\frac{1}{D}G_0(u,\xi)+G_1(u,\xi)+O(D). \tag{2.6}$$

Then G_0 will satisfy the problem (2.2) and (2.3), whose general solution is

$$G_0(u,\xi)=g_0(\xi)e^{(\xi-a)u}, \tag{2.7}$$

and this decays exponentially as $u\to\infty$ for $\xi<a$. We will determine $g_0(\xi)$ shortly, and also show that the expansion (2.6) must include the term of order $O(D^{-1})$ in order to match to the outer expansion.

Using (2.6) in (1.1) we find that the correction term G_1 satisfies

$$
\begin{aligned}
G_{1,uu}+(a-\xi)G_{1,u} &= -G_{0,\xi\xi}-(\xi G_0)_\xi \\
&= -e^{(\xi-a)u}\left[g_0''+(\xi+2u)g_0'+(u^2+u\xi+1)g_0\right]
\end{aligned}
\tag{2.8}
$$

and (1.2) yields

$$G_{1,u}(0,\xi)+(a-\xi)G_1(0,\xi)=0. \tag{2.9}$$

We write the solution to (2.8) as

$$G_1(u,\xi)=e^{(\xi-a)u}\overline{G}(u,\xi)+h(\xi). \tag{2.10}$$

Then the BC (2.9) yields

$$\overline{G}_u(0,\xi)+(a-\xi)h(\xi)=0. \tag{2.11}$$

From (2.8) and (2.10) we obtain

$$\overline{G}_{uu} + (\xi - a)\overline{G}_u = -\big[g_0'' + (\xi g_0)' + u(2g_0' + \xi g_0) + u^2 g_0\big]$$

so that

$$\overline{G}(u,\xi) = \alpha(\xi)\frac{u^3}{6} + \beta(\xi)\frac{u^2}{2} + \gamma(\xi)u + \delta(\xi) \tag{2.12}$$

where

$$\begin{aligned} \alpha(\xi) &= -\frac{2}{\xi - a}g_0(\xi), \quad \beta(\xi) = -\frac{2}{\xi - a}g_0'(\xi) + \left[\frac{2}{(\xi - a)^2} - \frac{\xi}{\xi - a}\right]g_0(\xi), \\ \gamma(\xi) &= -\frac{1}{\xi - a}\big[g_0'' + (\xi g_0)'\big] + \frac{1}{(\xi - a)^2}[2g_0' + \xi g_0] - \frac{2}{(\xi - a)^3}g_0 \end{aligned} \tag{2.13}$$

and $\delta(\xi)$ is an arbitrary function.

The asymptotic matching of the inner and outer solutions requires that as $u \to \infty$ (2.6) agrees with F_0 as $x \to 0^+$, and since $\xi < a$ this implies that

$$h(\xi) = F_0(0,\xi) = K\frac{1}{2\pi i}\int_{\mathrm{Br}} \frac{e^{-\xi^2/4}}{\sqrt{2\pi}} D_{a\theta + \theta^2}(\xi + 2\theta)A(\theta)d\theta. \tag{2.14}$$

Since $\overline{G}_u(0,\xi) = \gamma(\xi)$, (2.11) yields

$$\gamma(\xi) = (\xi - a)h(\xi) = \frac{d^2}{d\xi^2}\left(\frac{g_0}{a - \xi}\right) + \frac{d}{d\xi}\left(\frac{\xi g_0}{a - \xi}\right) \tag{2.15}$$

where the last equality follows from (2.13). We thus set $g_0(\xi) = (a - \xi)\overline{g}(\xi)$ to find that

$$\overline{g}''(\xi) + (\xi\overline{g})'(\xi) + (a - \xi)F_0(0,\xi) = 0. \tag{2.16}$$

Also, the leading term in the boundary layer becomes

$$G(u,\xi) \sim \frac{1}{D}(a - \xi)e^{(\xi - a)u}\overline{g}(\xi), \quad \xi < a. \tag{2.17}$$

By integrating (2.1) from $x = 0$ to $x = \infty$ and comparing the result to (2.15), we conclude that $\mathcal{D}(\xi) \equiv \int_0^\infty F_0(x,\xi)dx + \overline{g}(\xi)$ satisfies $\mathcal{D}''(\xi) + (\xi\mathcal{D})'(\xi) = 0$ for $\xi < a$, so we have

$$\overline{g}(\xi) + \int_0^\infty F_0(x,\xi)dx = \frac{L}{\sqrt{2\pi}}e^{-\xi^2/2}, \quad \xi < a \tag{2.18}$$

for some constant L. But the left side of (2.18) is the leading term in the expansion of the left side of (1.4) for $D \to 0^+$, hence $L = 1$. Note also that the normalization condition in (1.3) implies to leading order that

$$\int_{-\infty}^{a} \overline{g}(\xi)d\xi + \int_{-\infty}^{\infty}\int_0^\infty F_0(x,\xi)dx\,d\xi = 1. \tag{2.19}$$

It remains only to determine K. For $\xi > a$ there is no boundary layer contribution for $x = O(D)$ to (1.4) and thus

$$\int_0^\infty F_0(x,\xi)dx = K\frac{1}{2\pi i}\int_{\mathrm{Br}'} \frac{e^{-\xi^2/4}}{\sqrt{2\pi}} D_{a\theta+\theta^2}(\xi+2\theta)\frac{A(\theta)}{\theta}d\theta = \frac{1}{\sqrt{2\pi}}e^{-\xi^2/2} \tag{2.20}$$

where we restrict $-a < \mathrm{Re}(\theta) < 0$ on Br' (this allows us to perform the x-integration using (2.4) by simply replacing $e^{\theta x}$ by $1/\theta$). For $\xi > a$ we can close the integrating contour in the right half-plane, picking up the residue from the pole at $\theta = 0$. We conclude that $K = 1$, as $A(0) = 1$ and $D_0(\xi) = e^{-\xi^2/4}$.

To summarize, we have obtained the outer solution as (2.4) with $K = 1$ and the boundary layer as (2.17), with (2.18) and $L = 1$. The correction term in the boundary layer is given by G_1 in (2.6), with (2.10)-(2.14). The function $\delta(\xi)$ can ultimately be found by higher order matchings; this would require we compute the correction term F_1 in the outer solution. The analysis shows that for D small and $x > 0$ (or $x \approx 0$ with $\xi > a$) the shape of the density for the elliptic problem is, to leading order, the same as that for the parabolic one. However for x small and $\xi < a$ a boundary layer develops for the elliptic problem, and this takes the place of the boundary mass present in the parabolic problem.

2.3 Corner Layer

Form (2.17) we see that the leading term vanishes as $\xi \uparrow a$, indicating another non-uniformity in the asymptotics. We thus study how the boundary layer disappears as ξ increases through the critical value a, by introducing the scaling

$$\xi - a = D^{1/4}z, \qquad x = D^{3/4}X \tag{2.21}$$

with

$$F(x,\xi) = H(X,z) \sim D^\nu H_0(X,z) \tag{2.22}$$

where ν is a constant to be determined. From (1.1) and (1.2) we find that H_0 satisfies

$$H_{0,XX} - zH_{0,X} + H_{0,zz} = 0; \qquad X > 0, \quad -\infty < z < \infty \tag{2.23}$$

$$H_{0,X}(0,z) - zH_0(0,z) = 0, \quad -\infty < z < \infty. \tag{2.24}$$

We note that $u = D^{-1/4}X$ and $(\xi - a)u = zX$. Note also that going from the u to X scales represents a "thickening" of the boundary layer as $\xi \uparrow a$.

We seek solutions of (2.23) in the separable form $H_0 = e^{\tau X}\mathcal{H}(z;\tau)$, where $\tau \in \mathbb{C}$ is a separation constant. We find that $\mathcal{H}$ satisfies the Airy equation

$$\mathcal{H}_{zz} = (z\tau - \tau^2)\mathcal{H},$$

so that $\mathcal{H}$ will be proportional to $Ai\big(\tau^{1/3}(z-\tau)\big)$ (we expect H_0 to decay as $z \to \infty$, corresponding to the disappearance of the boundary layer). We thus argue that a general

solution to (2.23) will have the form

$$H_0(X,z) = \frac{1}{2\pi i}\int_C e^{\tau X} Ai(\tau^{1/3}z - \tau^{4/3}) f(\tau)d\tau \tag{2.25}$$

for some function f and contour C in the τ-plane. If the contour C can be chosen in the range $\mathrm{Re}(\tau) < 0$ then we integrate (2.23) and use (2.24) and (2.25) to obtain

$$\frac{d^2}{dz^2}\left\{\frac{1}{2\pi i}\int_C \frac{f(\tau)}{\tau} Ai(\tau^{1/3}z - \tau^{4/3})d\tau\right\} = 0, \tag{2.26}$$

so that the integral must be a linear function of z, and f satisfies a Fredholm integral equation of the first kind.

While we have not been able to determine $f(\tau)$ explicitly, we can show that (2.25) can asymptotically match with the outer solution in (2.4). For X and z large and positive we expect to have to scale τ in (2.25) to be small. If $f(\tau) \sim f_1\tau^{\nu_1}$ as $\tau \to 0$ the right side of (2.25) may be approximated by

$$\frac{1}{2\pi i}\int_C f_1\tau^{\nu_1} e^{\tau X} Ai(\tau^{1/3}z)d\tau. \tag{2.27}$$

We next examine (2.4) for $x \to 0$ and $\xi \to a$. For $x \to 0$ we must consider $\theta \to \infty$. In [4] we evaluated the infinite product in (2.4) as $\theta \to \infty$, with the result

$$A(\theta) \sim \frac{a}{\theta} e^{C_3} e^{-a^2/4}\sqrt{\theta} e^{-(\theta^2+a\theta)\log\theta} e^{\theta^2/2}. \tag{2.28}$$

Here $C_3 = C_3(a)$ is a constant. With the scaling $b \to \infty$, $Y \to \infty$ and $Y - 2\sqrt{b} = O(b^{-1/6})$, the parabolic cylinder function $D_b(Y)$ may be approximated by an Airy function:

$$D_b(Y) \sim 2^{b/2}\Gamma\left(\frac{b+1}{2}\right) b^{1/6} Ai\big(b^{1/6}(Y - 2\sqrt{b}\,)\big). \tag{2.29}$$

Applying (2.29) with $b = a\theta + \theta^2$ and $Y = \xi + 2\theta$ we have

$$(Y - 2\sqrt{b}\,)^{1/6} = [\xi + 2\theta - 2\sqrt{\theta^2 + a\theta}\,][a\theta + \theta^2]^{1/6} \sim (\xi - a)\theta^{1/3}$$

and then

$$D_{a\theta+\theta^2}(\xi + 2\theta) \sim \sqrt{2\pi}\theta^{1/3} e^{(1/2)(\theta^2+a\theta)\log(\theta^2+a\theta)} e^{-(1/2)(a\theta+\theta^2)} Ai\big((\xi - a)\theta^{1/3}\big). \tag{2.30}$$

This holds in the limit $\theta \to \infty$ with $\xi - a = O(\theta^{-1/3})$. Combining (2.28) and (2.30) in (2.4) and recalling that $K = 1$ leads to

$$\begin{aligned} F_0(x,\xi) &\sim \frac{1}{2\pi i}\int_{\mathrm{Br}} e^{\theta x} a e^{C_3} e^{-a^2/4} \frac{Ai((\xi - a)\theta^{1/3})}{\theta^{1/6}} d\theta \\ &= a e^{C_3} e^{-a^2/4}\left[\frac{1}{2\pi i}\int_{\mathrm{Br}} e^{\tau X} \frac{Ai(z\tau^{1/3})}{\tau^{1/6}} d\tau\right] D^{-5/8}. \end{aligned} \tag{2.31}$$

By comparing $D^{\nu}\times$ (2.27) with (2.31) we see that matching is possible with

$$\nu = -\frac{5}{8}, \ \nu_1 = -\frac{1}{6}, \ f_1 = ae^{C_3}e^{-a^2/4}. \tag{2.32}$$

Thus the density in this corner region will be $O(D^{-5/8})$. This means that the total mass in the corner is (noting that $dx\,d\xi = Ddz\,dX$) $O(D^{3/8})$.

Finally we investigate the matching between the corner and boundary layers. Noting that $\overline{g}(\xi)$ represents the mass in the boundary layer for a given $\xi < a$, we expect that $\overline{g} \to 0$ as $\xi \uparrow a$. Let us assume that

$$\overline{g}(\xi) \sim \overline{g}_2(a-\xi)^{\nu_2}, \qquad \xi \to a.$$

Then since $a - \xi = -D^{1/4}z$ the boundary layer becomes $O\big[D^{-1}(a-\xi)^{\nu_2+1}\big] = O[D^{(\nu_2-3)/4}]$ on the z scale. This can match to the corner layer if $(\nu_2-3)/4 = -5/8$, i.e., $\nu_2 = 1/2$. Now suppose we let $z \to -\infty$ and $X \to 0$ in (2.24), scaling $\tau = z + \omega(-z)^{-1/3}$. This yields

$$\begin{aligned} D^{-5/8}H_0(X,z) &= D^{-5/8}e^{zX}(-z)^{-1/3}\frac{1}{2\pi i}\int_{C'} e^{\omega X/(-z)^{1/3}} f(z+\omega(-z)^{1/3}) \\ &\quad \times Ai\big\{[-z-\omega(-z)^{-1/3}]^{1/3}\omega(-z)^{1/3}\big\}d\omega. \end{aligned} \tag{2.33}$$

If $f(z) \sim f_3(-z)^{\nu_3}$ as $z \to -\infty$ the left side of (2.33) should approach

$$D^{-5/8}e^{zX}f_3(-z)^{\nu_3-1/3}\left[\frac{1}{2\pi i}\int_{C'} Ai(\omega)d\omega\right]. \tag{2.34}$$

This can match to (2.16) if

$$\nu_3 = \frac{11}{6}, \qquad \overline{g}_2 = f_3\left[\frac{1}{2\pi i}\int_{C'} Ai(\omega)d\omega\right].$$

This shows that the matchings between the corner layer and other two expansions are possible if $f(\tau) = O(\tau^{-1/6})$ as $\tau \to 0$ and $f(\tau) = O\big(|\tau|^{11/6}\big)$ as $\tau \to -\infty$.

3 Conclusion

Our analysis showed that the density $F(x,\xi)$ is $O(1)$ for $x > 0$, and is large $(O(D^{-1}))$ in the boundary layer where $x = O(D)$ and $\xi < a$. It was also shown to be large $(O(D^{-5/8}))$ in the corner layer where $(x,\xi) \approx (0,a)$, with the precise scaling given by (2.21). The mass in the boundary layer and outer region are comparable, while that in the corner layer is asymptotically small. We compare this to the solution of the forward/backward problem in [4], where there is non-zero boundary mass along $x = 0$ and $\xi < a$.

We are presently investigating the problem (1.1)-(1.3) in the limit $a \to \infty$, with D fixed. This corresponds, after some rescaling, to having small diffusion in both the x and ξ directions. Some preliminary results show that a geometrical optics type expansion is fruitful, and that the problem yields interesting asymptotic structures, such as a caustic boundary and interior cusped caustics.

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